Extremal and Probabilistic Graph Theory
March 15

- **Mantel's Theorem.** Triangle-free graph with maximum number of edges is unique, which is $K_{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil}$
- **Proof.** By induction on n.

  Base case is trivial: when $n = 1, 2, 3$

  Let G be a $n-vertex$ K_3-free graph with at least $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges, we show that G must be $K_{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil}$

  Since adding a new edge to $K_{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil}$ will result in a K_3, by possibly deleting edges, we may assume G has exactly $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges.

  Since $e(G) = \left\lfloor \frac{n^2}{4} \right\rfloor$, $\delta(G) \leq \lfloor \frac{n}{2} \rfloor$, so there exist a vertex v with $d(v) \leq \lfloor \frac{n}{2} \rfloor$, let $G' = G - \{v\}$

  So G' is on $(n-1)-vertex$ K_3-free graph with $e(G') = e(G) - \lfloor \frac{n}{2} \rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor - \lfloor \frac{n}{2} \rfloor = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$

  By induction $G' = K_{\lfloor \frac{n-1}{2} \rfloor \lceil \frac{n-1}{2} \rceil}$ and $d(v) = \lfloor \frac{n}{2} \rfloor$

  Since G is $K_3 - free$, $N_G(v) \subset X$ or $N_G(v) \subset Y$

  when $n = 2k$, $N_G(v) = Y$

  when $n = 2k - 1$, $N_G(v) = Y(or\ X)$

  Then $\{X \cup \{v\}, Y\}$ gives a complete graph $K_{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil}$. ∎

- **Remak.** (1).$ex(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor$

  (2).$\pi(K_3) = \frac{1}{2}$

- **Def.** *Turán graph* $T_r(n)$: a balanced complete $r-partite$ $n-vertex$ graph:
- $V(G) = V_1 \dot\cup V_2 \dot\cup ... \dot\cup V_r$
- $|V_1| \leq |V_2| \leq ... \leq |V_r| \leq |V_1| + 1$
- all possible edges between $V_i \& V_j, i \neq j$
- **Observation.** (1). $e(T_r(n)) = \sum\limits_{0 \leq i < j < r} \lfloor \frac{n+i}{r} \rfloor \left\lfloor \frac{n+j}{r} \right\rfloor = \max\{e(G) : G$ is n-vertex r-partite $\}$

  (2). $T_r(n-1)$ can be obtained from $T_r(n)$ by deleting a vertex with min-degree in $T_r(n)$. i.e. a vertex with degree $n - \lceil \frac{n}{r} \rceil$.

  (3). $T_r(n)$ is the $n-vertex$ graph with highest min-degree among all graphs with the same number of edges $e(T_r(n))$.

- **Def.** A graph G is $r-partite$ if $V(G)$ can be partitioned into $V_1, V_2, ..., V_r$, s.t. each V_i is an independent set.

  $\Leftrightarrow$ $\chi(G) \leq r$

- **Turán Theorem.** Let G be on n-vertex $K_{r+1} - free$ graph. Then $e(G) \leq e(T_r(n))$ with equality if and only if $G = T_r(n)$.

- **Remark.** (1).$ex(n, K_{r+1}) = e(T_r(n))$

  (2). $\pi(K_{r+1}) = 1 - \frac{1}{r}$

- **Proofs.**There are 3 proofs in the following class notes:

  http://staff.ustc.edu.cn/ jiema/Comb2015/week

- **proof.** By induction on n.

  Let G be on $n - vertex\ K_{r+1} - free$ graph with at least $e(T_r(n))$ edges.

  we want to show $G = T_r(n)$, we may assume $e(G) = e(T_r(n))$

  By Observation (3), $\exists$ a vertex v in G with $d_G(v) \leq \delta(T_r(n))$,

  let $G' = G - \{v\}$ be $(n-1) - vertex\ K_{r+1} - free$ with

  $e(G') = e(T_r(n)) - \delta(T_r(n)) \overset{by\ obs\ (2)}{=} e(T_r(n-1))$,

  by induction, $e(G') = e(T_r(n))\&G' = T_r(n-1)$

  let $V(G') = V_1 \dot{\cup} V_2 \dot{\cup} ... \dot{\cup} V_r$, then $\exists$ some V_i(say V_1), $(V_1 \cup \{v\}, V_2, ..., V_r)$ is $r - partition$ of G with $e(G) = e(T_r(n))$.

  By Obs (1) and its unique maximality, $G = T_r(n)$. ∎

- **Def.** The chromatic number $\chi(G) = min\ k$ s.t. G can be partitioned into $V_1, V_2, ..., V_k$, where V_i is independent.

- **Note that:**$\chi(G) \leq k \Leftrightarrow$ G is $k - partite$.

- **Key Observation:** For large m, $T_r(m)$ contains every F with $\chi(F) \leq r$, but contains NO graph F with $\chi(F) \geq r + 1$.

  We also note that $T_r(n)$ is a $\frac{m}{r} - blowup$ of K_r.

- **Blowup Theorem.** (Recall)...

- **Theorem 1.** $\pi(T_r(m)) = \pi(K_r) = 1 - \frac{1}{r-1},\ for\ \forall m$

- **Theorem 2 (Erdős-Stone).** For $\forall$ graph F with $\chi(F) \geq 2,\ \pi(F) = 1 - \frac{1}{\chi(F)-1} = \pi(K_{\chi(F)})$

- **proof .** Let $\chi(F) = r$, then $\exists$ large m s.t. $F \subset T_r(m)$.

  $ex(n, F) \leq ex(n, T_r(n)) \Rightarrow \pi(n, F) \leq \pi(n, T_r(n)) \overset{Thm1}{=} 1 - \frac{1}{r-1}$

  since $\chi(F) = r,\ F \not\subset T_{r-1}(n) \Rightarrow ex(n, F) \geq 1 - \frac{1}{r-1}$,

  overall $\pi(F) = 1 - \frac{1}{r-1}$. ∎

- **Def.** Let $\mathcal{F}$ be a family of graphs.

  Let $\chi(\mathcal{F}) = min_{F \in \mathcal{F}} \chi(F)$.

- **Theorem 3 (Erdős-Stone; Observed by Simonovits).**
  For any family $\mathcal{F}$ with $\chi(\mathcal{F}) = r \geq 2$, we have $\pi(\mathcal{F}) = 1 - \frac{1}{r-1}$.
- **proof.** Exercise. ∎
- **Remak.** For any family of graphs $\pi(\mathcal{F}) \in \{0, \frac{1}{2}, \frac{2}{3}, ..., \frac{n-1}{n}, ...\}$.
- **Conjecture (Erdős-Simonovits).** For any rational $r \in [1, 2)$, there exists a bipartite graph F s.t. $ex(n, F) = \Theta(n^r)$.
- **Remak.** Recently it is proved that there exists a family $\mathcal{F}$ such that $ex(n, \mathcal{F}) = \Theta(n^r)$ by Bukh-Conlon. More details can be found in their paper: http://arxiv.org/pdf/1506.06406.pdf